

Triality in $SU(2)$ Seiberg-Witten theory and Gauss hypergeometric function

Ta-Sheng Tai*

Theoretical Physics Laboratory, RIKEN, Wako, Saitama 351-0198, JAPAN

Abstract

Through AGT conjecture, we show how triality observed in $\mathcal{N} = 2$ $SU(2)$ $N_f = 4$ QCD can be interpreted geometrically as the interplay among six of Kummer's twenty-four solutions belonging to one fixed Riemann scheme in the context of hypergeometric differential equations. We also stress that our presentation is different from the usual crossing symmetry of Liouville conformal blocks, which is described by the connection coefficient in the case of hypergeometric functions. Besides, upon solving hypergeometric differential equations at the zeroth order by means of the WKB method, a curve (thrice-punctured Riemann sphere) emerges. The permutation between these six Kummer's solutions then boils down to the outer automorphism of the associated curve.

1 Introduction

Adding N_f massless fundamental hypermultiplets (flavors) to pure $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory results in $SO(2N_f)$ flavor symmetry which gets enhanced to $Spin(2N_f)$ at the quantum level. This is due to the fact that monopoles in the low-energy Coulomb phase transform as spinors of $Spin(2N_f)$ once $2N_f$ fermionic zero-modes (collective coordinate) on them are quantized. In particular, when $2N_c = N_f = 4$ where both vanishing one-loop β -function and exact scale invariance follow one sees that the outer automorphism group \mathbf{S}_3 of $Spin(8)$, a double-cover of $SO(8)$, gets realized as a kind of S -duality, i.e. *triality*. Namely, its action in Coulomb phase permutes three fundamental BPS objects corresponding to three eight-dimensional irreducible representations of $SO(8)$, i.e. $(\mathbf{v}, \mathbf{s}, \mathbf{c}) \equiv (\text{electron}, \text{monopole}, \text{dyon})$. Our aim in this short letter is to clarify its geometric origin in terms of Gauss hypergeometric functions.

In [1] Seiberg and Witten wrote down explicitly the $SU(2)$ $N_f = 4$ Seiberg-Witten (SW) curve Σ parameterized by four bare flavor masses and $\tau_0 \equiv \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$ (marginal bare gauge coupling)¹. They discovered the following exotic transformation rule attributable to the property of triality:

$$S: \tau_0 \rightarrow -\frac{1}{\tau_0}, \quad \begin{cases} m_1 \rightarrow \frac{1}{2}(m_1 + m_2 + m_3 - m_4) \\ m_2 \rightarrow \frac{1}{2}(m_1 + m_2 - m_3 + m_4) \\ m_3 \rightarrow \frac{1}{2}(m_1 - m_2 + m_3 - m_4) \\ m_4 \rightarrow \frac{1}{2}(-m_1 + m_2 + m_3 - m_4) \end{cases} \quad (1.1)$$

* e-mail address: tasheng@riken.jp

¹ In the presence of fundamental flavors, τ_0 's normalization deviates from the pure $\mathcal{N} = 2$ one by a factor 2.

$$T : \tau_0 \rightarrow \tau_0 + 1, \quad \begin{cases} m_1 \rightarrow m_1 \\ m_2 \rightarrow m_2 \\ m_3 \rightarrow m_3 \\ m_4 \rightarrow -m_4 \end{cases} \quad (1.2)$$

under which Σ is kept invariant. This is because combinations of S and T together generate $SL(2, \mathbb{Z})/\Gamma(2) = \{I, S, T, ST, TS, STS\}$ which is identical to the outer automorphism group \mathbf{S}_3 . Notice that these rules arising from observation still lack rigorous derivation. It is then seen that full $SL(2, \mathbb{Z})$ invariance w.r.t. τ_0 shrinks to a smaller $\Gamma(2)$ one by including flavor mass deformations unless (1.1) and (1.2) are taken into account. Explaining triality in a more geometric way has been attempted ever since $\mathcal{N} = 2$ $SU(2)$ low-energy Coulomb phase dynamics got rephrased in, say, Vafa's F-theory setup [2, 3] or Gaiotto's picture. Either one seems promising because totally geometric reformulations of a class of $\mathcal{N} = 2$ theories were provided there.

The latter due to Gaiotto's seminal paper [4] interpreted nicely S -duality group of a large family of $\mathcal{N} = 2$ superconformal quiver gauge theories as physically equivalent ways of deforming certain genus- g n -punctured Riemann surface $C_{g,n}$ ². Also, $C_{g,n}$'s moduli (Teichmüller) space is accordingly identified with the space of a set of ultra-violet coupling constants, say, $q_{UV} = e^{\pi i \tau_{UV}}$. Based on his idea, the previous \mathbf{S}_3 has to be thought of as $\mathbf{S}_4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ ($\mathbb{Z}_2 \times \mathbb{Z}_2$: Klein four-group or Vierergruppe) which manages to permute marked punctures on $C_{0,4}$ responsible for the $2N_c = N_f = 4$ case. More precisely, from two $SO(4)$ ($SO(4) \times SO(4) \sim SU(2)_a \times SU(2)_b \times SU(2)_c \times SU(2)_d$) of $SO(8)$ one can decompose $\mathbf{8}$ as

$$\mathbf{8} \sim (\mathbf{2}_a \otimes \mathbf{2}_b) \oplus (\mathbf{2}_c \otimes \mathbf{2}_d);$$

therefore the action of triality exchanging three $\mathbf{8}$'s results in permuting punctures labeled by $SU(2)_\xi$ ($\xi = a, b, c, d$) respectively. As pointed out later by Alday, Gaiotto and Tachikawa [5]³, further associating every puncture with a mass parameter μ subject to

$$\begin{aligned} m_1 &= \mu_a + \mu_d - \frac{Q}{2}, & m_2 &= -\mu_a + \mu_d + \frac{Q}{2}, \\ m_3 &= \mu_c + \mu_b - \frac{Q}{2}, & m_4 &= -\mu_c + \mu_b + \frac{Q}{2}, \end{aligned} \quad (1.3)$$

one easily agrees that (1.1) and (1.2) can be completely accounted for by permutations of μ 's with $Q = 0$. Henceforth, (1.3) nowadays referred to as “AGT dictionary” opens up a new perspective for understanding triality geometrically⁴. In fact, by introducing F-theoretically a vev of an $SO(8)$ adjoint scalar field Φ living on D7-branes, say,

$$\langle \Phi \rangle = \begin{pmatrix} i\sigma_2 m_1 & & & \\ & i\sigma_2 m_2 & & \\ & & i\sigma_2 m_3 & \\ & & & i\sigma_2 m_4 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (1.4)$$

μ 's (when $Q = 0$) just stand for its diagonal Cartan elements w.r.t. $SO(4) \times SO(4)$ decomposition. In addition, μ 's get related to momenta of 2D Liouville primary fields $V_\mu = e^{2\mu\varphi}$ (φ : Liouville field)

²Though at first sight $C_{g,n}$ seems an ultra-violet object, a r -sheeted cover of it turns out to be the infra-red Seiberg-Witten curve of A_{r-1} -type $SU(r)$ SCFTs.

³See also [6]-[57] for recent developments along AGT conjecture.

⁴See [58] for another geometric interpretation of triality resulting from E-string formalism.

with a conformal dimension $\Delta(\mu) = \mu(Q - \mu)$. A zero background charge $Q = b + \frac{1}{b} = \frac{\epsilon_1 + \epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}} = 0$ required here corresponds to a 4D physical field theory limit $\epsilon_1 = -\epsilon_2$ ⁵.

While these arguments do render a satisfactory explanation of the geometric origin of triality, we instead would like to explore another possibility using Gauss hypergeometric functions. What makes this accessible is again due to AGT conjecture which proposes an equivalence between a 2D Liouville conformal block \mathcal{B} defined on $C_{g,n}$ and the instanton part of Nekrasov's partition function $Z_{\text{inst}}[C_{g,n}]$ of a 4D $\mathcal{N} = 2$ A_1 -type SCFT. Under special circumstances, the four-point spherical $\mathcal{B}[C_{0,4}]$ satisfies a hypergeometric differential equation (HDE). Therefore, based on the equality $Z_{\text{inst}}[C_{0,4}] = \mathcal{B}[C_{0,4}]$ with q_{UV} regarded as the cross-ratio of four punctures, one can interpret (1.1) and (1.2) as interchanging solutions of a HDE fixed by some Riemann scheme because $Z_{\text{inst}}(\mathbf{a}, \vec{m}, q_{UV}(\tau_0), \epsilon_1, \epsilon_2)$ (\mathbf{a} : Coulomb phase parameter) itself contains the exact solution (SW curve) to infra-red dynamics. To conclude, we find that triality generates six out of Kummer's twenty-four solutions. By grouping them properly into three pairs, each pair just spans the basis of solutions belonging to respectively $(0, 1, \infty)$ known as regular singularities of a second-order HDE.

This letter is organized as follows. In section 2, we review necessary aspects about hypergeometric functions. Especially, the elliptic *lambda* function relating τ_{UV} to τ_0 will play a quite profound role in latter discussions. In section 3, we show how triality can be interpreted as the interplay among six of Kummer's twenty-four solutions via AGT conjecture. A summary is given in section 4.

2 Preliminaries

2.1 Hypergeometric function

Let us first recall some main features of Gauss hypergeometric functions and their relation to the modular curve $X_2 = \mathbb{H}/\Gamma(2)$ being isomorphic to $\mathbb{C} \setminus \{0, 1\}$. See for example [59] for details. Here, X_N is in general a noncompact Riemann surface whilst \mathbb{H} stands for the upper half-plane. $\Gamma(N)$ denotes the level N principal congruence subgroup of $SL(2, \mathbb{Z})$:

$$\Gamma(N) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}, \quad ad - bc = 1.$$

Getting familiar with these stuffs serves as our cornerstone of clarifying the role of triality observed in the $2N_c = N_f = 4$ SW theory by means of Gauss hypergeometric functions $y(z)$, solutions of a second-order linear ODE:

$$z(1-z)y(z)'' + (c - (a+b+1)z)y(z)' - aby(z) = 0, \quad z \in \mathbb{C}. \quad (2.1)$$

Meanwhile, there are three *regular* singularities $(0, 1, \infty)$ near each of which two linearly independent solutions to (2.1) exist⁶. That is, at $z = 0$

$$\begin{cases} y_{01} = {}_2F_1(a, b, c; z), \\ y_{02} = z^{1-c} {}_2F_1(a - c + 1, b - c + 1, 2 - c; z); \end{cases}$$

⁵ $\epsilon_{1,2}$ are related to the size of a unit rectangle in Young tableaux appearing in Nekrasov's partition functions.

⁶At points other than $(0, 1, \infty)$, (2.1) can be simplified to $y(z)'' = 0$ by changes of variables.

at $z = 1$

$$\begin{cases} y_{11} = {}_2F_1(a, b, a + b - c + 1; 1 - z), \\ y_{12} = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b, c - a - b + 1; 1 - z); \end{cases}$$

at $z = \infty$

$$\begin{cases} y_{\infty 1} = (-z)^{-a} {}_2F_1(a, a - c + 1, a - b + 1; z^{-1}), \\ y_{\infty 2} = (-z)^{-b} {}_2F_1(b, b - c + 1, b - a + 1; z^{-1}). \end{cases}$$

According to the local exponent of z around each singularity, three pairs of solutions listed above can be summarized by Table 1 (Riemann scheme) in the context of Fuchsian linear differential equations. Also, due to Fuchs relation summing up all entries inside the last two rows of Table 1 gives zero. As a matter of fact, each pair of solutions can be transformed to one another through

Table 1: Riemann scheme

$z = 0$	$z = 1$	$z = \infty$
0	0	a
$1 - c$	$c - a - b$	b

suitable two by two matrices (connection coefficients); for instance,

$$(y_{01}, y_{02}) = (y_{11}, y_{12})P_{01},$$

$$P_{01} = \begin{pmatrix} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} & \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \end{pmatrix}. \quad (2.2)$$

Furthermore, $y(z)$ suffers monodromies like

$$(y_{\ell 1}, y_{\ell 2}) \rightarrow (y_{\ell 1}, y_{\ell 2})M, \quad M \in \pi_1(\mathbb{C} \setminus \{0, 1\})$$

when winding around each singularity. That there exists a group homomorphism between the fundamental group $\pi_1(\mathbb{C} \setminus \{0, 1\})$ and $GL(2, \mathbb{C})$ leads to $M(\gamma, z_0) \in GL(2, \mathbb{C})$ (modulo conjugation) for each path γ given a reference point z_0 . Due to $\gamma_1 \cdot \gamma_2 = \gamma_\infty$ one is able to establish $M(\gamma_1, z_0) \cdot M(\gamma_2, z_0) = M(\gamma_\infty, z_0)$ where γ_1 (γ_2) is designated to surround the singularity $z = 0$ ($z = 1$) counterclockwise.

2.2 Schwarz map

Next, we proceed to consider the ratio $\mathcal{D}_\ell = \frac{y_{\ell 1}}{y_{\ell 2}}$ defining the famous triangle Schwarz map, a special case of conformal Schwarz-Christoffel maps which bring the upper half-plane \mathbb{H} to certain n -vertex polygon. In fact, the setup under consideration can be cast into the so-called uniformization problem for the simplest case— a three-punctured sphere $\mathbb{C} \setminus \{0, 1\}$. One can arrange (2.1) into a Q -form

$$\partial_z^2 y + \frac{1}{2} \{\rho, z\} y = 0, \quad \{\rho, z\} : \text{Schwarzian derivative of } \rho, \quad \rho = \frac{y_\varrho}{y_\varsigma}.$$

There, the multi-valued ρ (ratio of two independent solutions) induces a map $\mathbb{C} \setminus \{0, 1\} \rightarrow (\text{unit disc}) / \mathcal{G}$ with branching points $(0, 1, \infty)$. $\mathcal{G} \subset SU(1, 1)$ denotes the monodromy group of ρ as will soon be seen.

While one takes $\rho = \mathcal{D}_\ell$, it naively maps \mathbb{H} to a triangle on a Riemann sphere \mathbb{P}^1 bounded by circular arcs. Connection coefficients P 's can thus be thought of as applying Möbius transformations (automorphism group of \mathbb{P}^1) to the triangle. Meanwhile, $(0, 1, \infty)$ on \mathbb{H} are brought to three vertices whose angles are $\pi\nu_\ell$ respectively:

$$\nu_0 = 1 - c = \frac{1}{p}, \quad \nu_1 = c - a - b = \frac{1}{q}, \quad \nu_\infty = b - a = \frac{1}{r}. \quad (2.3)$$

(p, q, r) are natural numbers greater than one. The relation between (2.3) and Table 1 can be made clear if one looks into the local behavior of \mathcal{D}_ℓ near each responsible singularity:

$$\mathcal{D}_0 \simeq z^{\nu_0} (1 + \mathcal{O}(z)), \quad \mathcal{D}_1 \simeq (1 - z)^{\nu_1} (1 + \mathcal{O}(1 - z)), \quad \mathcal{D}_\infty \simeq z^{-\nu_\infty} (1 + \mathcal{O}(z^{-1})).$$

Of course, extending $\mathcal{D}_\ell(\mathbb{H})$ to $\mathcal{D}_\ell(\mathbb{C})$ is totally possible and one encounters

$$\mathcal{D}_\ell \rightarrow \frac{a\mathcal{D}_\ell + b}{c\mathcal{D}_\ell + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}) \quad (2.4)$$

just because of monodromies when winding around each responsible singularity. The resulting image $\mathcal{D}_\ell(\mathbb{C})$ becomes two sets of triangles⁷ which tile the entire \mathbb{P}^1 if (2.3) is assumed. Certainly, after an automorphism (pattern-preserving) group $\Gamma(p, q, r)$ is divided, one is able to claim that

$$\mathcal{D}_\ell : \mathbb{C} \setminus \{0, 1\} \rightarrow (\mathbb{P}^1 - \text{triangle vertices}) / \Gamma(p, q, r). \quad (2.5)$$

As will be justified below, in view of (2.4) we cannot help but regard \mathcal{D}_ℓ as the complex moduli of some elliptic curve $E_{\mathcal{D}_\ell}$ with $\mathcal{D}_\ell \in \mathbb{H}/g$ (modular curve). This way of thinking is also inspired by the definition of \mathcal{D}_ℓ being a ratio of two hypergeometric functions both of which solve Fuchsian equations and are identified with period integrals over an algebraic curve. That generators of g must be those of $\pi_1(\mathbb{C} \setminus \{0, 1\}, z_0)$ w.r.t. \mathcal{D}_ℓ confirms that there exists a group homomorphism between $\pi_1(\mathbb{C} \setminus \{0, 1\}, z_0)$, $\Gamma(p, q, r)$ and g . Indeed, we will find that the above picture is realized when $(p, q, r) = (\infty, \infty, \infty)$ and $g = \Gamma(2)$.

2.3 Elliptic curve and λ -function

Let us proceed to clarify the appearance of an elliptic curve $E_{\mathcal{D}}$ mentioned above. An integral representation of ${}_2F_1(a, b, c; z)$ for $z \neq (0, 1, \infty)$ is given by⁸

$$\int_{\gamma} u^{-\mu_0} (u - 1)^{-\mu_1} (u - z)^{-\mu_z} du = \int_{\gamma} \eta(z), \quad \mu_0 + \mu_1 + \mu_z + \mu_\infty = 2 \quad (2.6)$$

⁷For instance, one can paint each triangle certain color according to which one of two half-planes they come from.

⁸Usual normalizations like Beta factors are omitted. Note also that with $(z_1, z_2, z_3, z_4) = (0, 1, z, \infty)$ a simplification occurs, i.e.

$$\int_{\gamma} \prod_{i=1}^4 (u - z_i)^{-\mu_i} du \rightarrow \int_{\gamma} \prod_{i=1}^3 (u - z_i)^{-\mu_i} du$$

due to the term involving $z_4 = \infty$ dropped.

where all μ 's are simple linear combinations of (a, b, c) and assumed to be rational. $\eta(z) \equiv \frac{du}{x}$ is defined w.r.t. an algebraic curve

$$X : x^\kappa = u^{\kappa\mu_0}(u-1)^{\kappa\mu_1}(u-z)^{\kappa\mu_z} \quad (2.7)$$

with κ being the least common denominator of μ 's. γ known as Pochhammer's contour now becomes some homology cycle of X , i.e. $\gamma \in H_1(X, \mathbb{Z})$. Inequivalent γ 's will lead to independent hypergeometric functions. X turns out to be an elliptic curve of the standard Legendre form: $x^2 = 4u(u-1)(u-z)$ when $\kappa = 2$ and $\mu_0 = \mu_1 = \mu_z = \mu_\infty = \frac{1}{2}$. This soon implies $(a, b, c) = (\frac{1}{2}, \frac{1}{2}, 1)^9$ from the parameterization:

$$\begin{cases} \mu_0 = \frac{1}{2}(1 - \nu_0 + \nu_1 - \nu_\infty), & \mu_1 = \frac{1}{2}(1 + \nu_0 - \nu_1 - \nu_\infty), \\ \mu_z = \frac{1}{2}(1 - \nu_0 - \nu_1 + \nu_\infty), & \mu_\infty = \frac{1}{2}(1 + \nu_0 + \nu_1 + \nu_\infty). \end{cases}$$

Now, consider the ratio

$$\tilde{\mathcal{D}}(z) = \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; z)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 1-z)} = \frac{K(\sqrt{z})}{K'(\sqrt{z})}, \quad K'(\sqrt{z}) = K(\sqrt{1-z}) \quad (2.8)$$

where K denotes the complete elliptic integral of the first kind. Conventionally, $\tilde{\mathcal{D}}(z)/2$ is called the *aspect* ratio of a rectangle yielded by performing a Schwarz-Christoffel map over \mathbb{H} . Through defining

$$\tau \equiv i\tilde{\mathcal{D}}(z) = \frac{\int_{\gamma_1} \eta(z)}{\int_{\gamma_2} \eta(z)}, \quad \gamma \in H_1(X(\mu_i), \mathbb{Z}), \quad \forall \mu_i = 1/2,$$

the famous isomorphism:

$$i\tilde{\mathcal{D}} : \mathbb{C} \setminus \{0, 1\} \rightarrow \mathbb{H}/\Gamma(2) \quad (2.9)$$

is induced. In particular, the appearance of $\Gamma(2)$ is due to the choice of (a, b, c) as explained around (2.12). τ becomes exactly the complex moduli of a torus \mathbb{C}/Λ_τ ($\Lambda_\tau \equiv \mathbb{Z}\tau + \mathbb{Z}$) which is isomorphic to X in (2.7) with $\forall \mu_i = 1/2$. In addition, the inverse of $\tilde{\mathcal{D}}(z)$ is known as the elliptic *lambda* function:

$$\lambda \equiv z = \frac{\theta_2^4(q)}{\theta_3^4(q)} \quad (2.10)$$

where $\theta_i(q)$'s are theta constants whilst $q = e^{i\pi\tau} = e^{-\pi\tilde{\mathcal{D}}}$ is called the *nome*. From now on, we will not especially distinguish between λ and z which eventually represent the cross-ratio of four points on \mathbb{P}^1 ¹⁰. By definition λ should be invariant under $\Gamma(2)$ or, equivalently, subject to

$$\lambda(\tau + 2) = \lambda(\tau), \quad \lambda\left(\frac{\tau}{1-2\tau}\right) = \lambda(\tau).$$

⁹Equivalently, $\nu_0 = \nu_1 = \nu_\infty = 0$ or $p = \lambda = r = \infty$.

¹⁰The cross-ratio of four points on \mathbb{P}^1 is given by

$$\lambda = (x_2, x_1; x_3, x_4) = \frac{(x_2 - x_3)(x_1 - x_4)}{(x_1 - x_3)(x_2 - x_4)}.$$

Furthermore, since (2.9) is a bijective map what is inferred is that six distinct λ 's define the same elliptic curve because of same Klein's absolute j -invariants they will provide. Namely, a six-to-one relation does follow owing to

$$j = \frac{4}{27} \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2} = \frac{g_2^3}{g_2^3 - 27g_3^2} \quad (2.11)$$

where g_2 and g_3 are modular invariants of an elliptic curve expressed in Weierstrass form:

$$y^2 = 4x^3 - g_2x - g_3$$

whose three distinct roots are $(e_1, e_2, e_3) := \left(\wp(\frac{1}{2}), \wp(\frac{\tau}{2}), \wp(\frac{\tau}{2} + \frac{1}{2})\right)$. Notice that \wp is Weierstrass's doubly-periodic function. Consequently,

$$\lambda = \frac{\wp(\frac{\tau}{2} + \frac{1}{2}) - \wp(\frac{1}{2})}{\wp(\frac{\tau}{2}) - \wp(\frac{\tau}{2} + \frac{1}{2})}.$$

We are led to the following conclusion. Upon defining $\text{Hom}(\pi_1(\mathbb{C} \setminus \{0, 1\}, z_0), SL(2, \mathbb{Z}))$, because generators of the monodromy group w.r.t. ${}_2F_1(a, b, c; z)$ are determined by (a, b, c) as

$$M_1 = \begin{pmatrix} 1 & 0 \\ -1 + e^{-2\pi ib} & e^{-2\pi ic} \end{pmatrix} \quad \text{and} \\ M_2 = \begin{pmatrix} 1 & 1 - e^{-2\pi ia} \\ 0 & e^{-2\pi i(a+b-c)} \end{pmatrix} \quad \text{with} \quad (a, b, c - a, c - b) \notin \mathbb{Z},$$

when $(a, b, c) = (\frac{1}{2}, \frac{1}{2}, 1)$ they are just those of $\Gamma(2)$, i.e.

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad (2.12)$$

as used in (2.9). As a remark, the relation (2.8) is completely not new since it has long been known as the infra-red gauge coupling τ_{IR} in the pure $SU(2)$ SW theory if one equates $(2 - z)/z$ with its Coulomb phase parameter there.

All in all, we have just wandered quite a lot from the conventional interpretation of (2.9), i.e. one can always express an elliptic curve in terms of a two-sheeted cover of a sphere with branching points $(0, 1, \lambda, \infty)$ such that the equivalence between their moduli spaces naturally introduces the underlying isomorphism (2.9) or its inverse- λ -function (2.10).

3 Triality in $SU(2)$ $N_f = 4$ Seiberg-Witten theory

A standard $2N_c = N_f = 4$ SW curve is of a rather sophisticated form parameterized by four bare flavor masses and τ_0 [1]:

$$y^2 = 4 \left[W_1 W_2 W_3 + A(W_1 T_1(e_2 - e_3) + W_2 T_2(e_3 - e_1) + W_3 T_3(e_1 - e_2)) - A^2 N \right] \quad (3.1)$$

where (u : Coulomb phase parameter)

$$W_i = x - e_i u - e_i^2 R, \quad A = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1)$$

and

$$\begin{aligned}
R &= \frac{1}{2} \sum_{i=1}^4 m_i^2, & N &= \frac{3}{16} \sum_{i>j>k} m_i^2 m_j^2 m_k^2 - \frac{1}{96} \sum_{i \neq j} m_i^2 m_j^4 + \frac{1}{96} \sum_{i=1}^4 m_i^6, \\
T_1 &= \frac{1}{12} \sum_{i>j} m_i^2 m_j^2 - \frac{1}{24} \sum_{i=1}^4 m_i^4, & T_2 &= -\frac{1}{2} \prod_{i=1}^4 m_i - \frac{1}{24} \sum_{i>j} m_i^2 m_j^2 + \frac{1}{48} \sum_{i=1}^4 m_i^4, \\
T_3 &= \frac{1}{2} \prod_{i=1}^4 m_i - \frac{1}{24} \sum_{i>j} m_i^2 m_j^2 + \frac{1}{48} \sum_{i=1}^4 m_i^4.
\end{aligned}$$

e_i 's are functions of τ_0 :

$$e_1 = \frac{1}{12}(\theta_3^4 + \theta_4^4), \quad e_2 = \frac{1}{12}(\theta_2^4 - \theta_4^4), \quad e_3 = \frac{1}{12}(-\theta_2^4 - \theta_3^4).$$

Here, τ_0 must be regarded as the asymptotic value of $\tau_{IR} = \tau_0 + \frac{1}{2\pi i} \left(\sum_{i=1}^4 \log(u - m_i^2) - 4 \log u \right) + \dots$ expanded at large u . Its reduction to asymptotically-free counterparts ($N_f \leq 3$) is easily achieved via tuning τ_0 and m_i in order to yield a suitable dynamical scale Λ_{N_f} .

Seiberg and Witten found that (3.1) is invariant under elements of

$$SL(2, \mathbb{Z})/\Gamma(2) = \mathbf{S}_3 = \{I, S, T, ST, TS, STS\} \quad (3.2)$$

if and only if (1.1) and (1.2) are taken into account simultaneously. This phenomenon is often referred to as triality whose origin may be owing to the outer automorphism group \mathbf{S}_3 of $Spin(8)$, the quantum flavor symmetry in the superconformal case ($m_i = 0$). Because we want to interpret triality as interchanging Kummer's solutions, our strategy is to think of $\Gamma(1)/\Gamma(2) = \mathbf{S}_3$ here as $\mathbf{S}_4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ on a four-punctured \mathbb{P}^1 via λ -function introduced in (2.10)¹¹.

In other words, under \mathbf{S}_3 τ_0 enlarges its “fundamental” domain¹² instead to $\mathbb{H}/\Gamma(2)$ and by (2.10) we will now map it bijectively to λ -space defined on $C_{0,4}$ where six distinct cross-ratios are caused by applying $\mathbf{S}_4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. That is, the action of \mathbf{S}_3 is translated to interchanging four marked punctures. The next step is to know how (1.1) and (1.2) can be incorporated into the four-point spherical conformal block $\mathcal{B}[C_{0,4}]$ with

$$\mathcal{B}(\alpha, \vec{\mu}, Q|\lambda) = Z_{inst}^{2N_c=N_f=4}(\mathbf{a}, \vec{m}, \epsilon_{1,2}|q_{UV}), \quad \alpha : \text{internal momentum} \quad (3.3)$$

where in addition to $\lambda \equiv q_{UV}(\tau_0)$ the dictionary between parameters on two sides has been spelt out by AGT. All in all, making use of properties of Gauss hypergeometric functions we will arrive at a new unifying understanding of this mysterious part of S -duality–triality for $2N_c = N_f = 4$.

3.1 Gaiotto's picture and AGT conjecture

Gaiotto's idea arises from rearranging old SW curves and leads to another way of engineering a huge class of $\mathcal{N} = 2$ $SU(r)$ SCFTs by wrapping r M5-branes on $C_{g,n}$, i.e. compactifying 6D A_{r-1} -type (2,0) theories on $C_{g,n}$ accompanied by a partial twisting. There are various ways of decomposing $C_{g,n}$ into trinions and tubes so S -duality (mapping class) group gets identified with

¹¹In AGT's Appendix (B.29) λ -function has already shown up.

¹²As (3.1) reduces to merely an usual Weierstrass elliptic curve characterized by τ_0 when all $m_i = 0$, so basically $\tau_0 \in \mathbb{H}/SL(2, \mathbb{Z})$.

such physically equivalent surgeries. In addition, weak-coupling limits are attained intuitively by elongating extremely tubes joining two punctures. This kind of *degenerate* limits correspond to *cusps* in the moduli space of $C_{g,n}$. Total $3g-3+n$ tubes contained in $C_{g,n}$ correspond to the number of gauge groups of a weakly-coupled quiver SCFT equipped with a Lagrangian description.

Let us elaborate arguments about aforementioned λ -space on $C_{0,4}$. Upon viewing λ as the coordinate on $\mathbb{P}^1 \setminus (0, 1, \infty)$ (up to a Möbius transformation), six distinct values generated from it by $\Gamma(1)/\Gamma(2) = \mathbf{S}_3$ are referred to as six different cross-ratios:

$$\begin{array}{cccccc} \text{element in } \mathbf{S}_3 & I & T & S & ST & TS & STS \\ \text{cross-ratio} & \lambda & \frac{\lambda}{\lambda-1} & 1-\lambda & \frac{1}{1-\lambda} & \frac{\lambda-1}{\lambda} & \frac{1}{\lambda} \end{array} \quad (3.4)$$

(3.4) can be derived directly based on either footnote 10 or (2.10) with modular properties of theta constants listed below:

$$\begin{aligned} \theta_2(q) \equiv \vartheta_{10}(0, \tau) &= \frac{1}{\sqrt{-i\tau}} \vartheta_{01}(0, -\frac{1}{\tau}), & \theta_3(q) \equiv \vartheta_{00}(0, \tau) &= \frac{1}{\sqrt{-i\tau}} \vartheta_{00}(0, -\frac{1}{\tau}), \\ \theta_4(q) \equiv \vartheta_{01}(0, \tau) &= \vartheta_{00}(0, \tau+1), & \vartheta_{10}(0, \tau) &= e^{-\frac{i\pi}{4}} \vartheta_{10}(0, \tau+1). \end{aligned}$$

Replacing $\mathbb{H}/\Gamma(2)$ by $(\mathbb{H}/SL(2, \mathbb{Z})) \times \mathbf{S}_3$ in (2.9) and recalling that another famous isomorphism

$$j : \mathbb{H}/SL(2, \mathbb{Z}) \rightarrow \mathbb{C} \setminus \{0, 1\}$$

is induced by Klein's j -invariant, one agrees that the identity (2.11) between j and λ describes a six-to-one relation.

Next, we focus on $\mathcal{B}[C_{0,4}]$ in (3.3). In general, because it satisfies Zamolodchikov's recursion relation [60] an expansion over λ to any desired order is possible. However, a closed-form expression of it is still missing. Nevertheless, if one of four inserted primary fields becomes degenerate, say, $V_{\mu_3} = \Phi_{2,1}$ it is well-known [61, 62] that by means of the null-state condition:

$$\left(L_{-2} - \frac{3}{2(2\Delta(h_{2,1}) + 1)} L_{-1}^2 \right) \Phi_{2,1} = 0, \quad h_{r,s} = \frac{1-r}{2}b + \frac{1-s}{2b}, \quad \mu_3 \equiv h_{2,1} = -\frac{b}{2},$$

one is led to

$$\mathcal{B}[C_{0,4}] \equiv \langle V_{\mu_1}(0) V_{\mu_2}(1) V_{\mu_3}(\lambda) V_{\mu_4}(\infty) \rangle = \lambda^{b\mu_1} (1-\lambda)^{b\mu_2} {}_2F_1(a, b, c; \lambda) \quad (3.5)$$

where¹³

$$\begin{cases} a = -N, \\ b = \frac{1}{\beta} \left(-\frac{2\mu_1}{\epsilon_1} - \frac{2\mu_2}{\epsilon_1} + 2 \right) + N - 1, \\ c = \frac{1}{\beta} \left(-\frac{2\mu_1}{\epsilon_1} + 1 \right), \\ N = -\epsilon_1(\mu_1 + \mu_2 + \mu_3 - \mu_4), \quad \beta = -\frac{\epsilon_2}{\epsilon_1}, \quad \epsilon_1 = b, \quad \epsilon_2 = \frac{1}{b}. \end{cases} \quad (3.6)$$

¹³We adhere to conventions used in [28]. Also, we wish “ b ” used in both ${}_2F_1(a, b, c; z)$ and Liouville theory side will cause no confusion.

The internal momentum α is set to be

$$\alpha = \frac{Q}{2} + \mathbf{a} = \mu_4 + \frac{b}{2}, \quad Q = b + \frac{1}{b}.$$

Adopting a 4D physical field theory limit $\epsilon_1 + \epsilon_2 = 0$ ($\beta = 1$) may give rise to a further simplification¹⁴. Note that N is designated to characterize the size of a *hermitian* matrix appearing in the recent Dijkgraaf-Vafa proposal [43]. There, an (A_1 -type) n -point spherical \mathcal{B} was rewritten in terms of a Penner-type matrix integral (or Selberg-Kaneko integral [63]). As a remark, from (3.6) ${}_2F_1(a, b, c; \lambda)$ also stands for a Jacobi polynomial defined by

$$G_N(\xi, \zeta; \lambda) = {}_2F_1(-N, \xi + N, \zeta; \lambda) = 1 + \sum_{r=1}^N (-)^r {}_N C_r \frac{\Gamma(\xi + N + r)\Gamma(\zeta)}{\Gamma(\xi + N)\Gamma(\zeta + r)} \lambda^r,$$

for $\zeta \neq 0, -1, -2, \dots, -N + 1$.

This sounds quite consistent with the fact pointed out by Schiappa and Wyllard [28] that a three-point DV matrix model Z_{3pt}^{DV} for $\mathcal{T}_{0,3}(A_1)$ is exactly solved by its orthogonal polynomial–Jacobi polynomial; namely, $\langle \det(\lambda - M) \rangle_{Z_{3pt}^{DV}}$ is equal to (3.5) without the factor $\lambda^{b\mu_1}(1 - \lambda)^{b\mu_2}$ as shown in [28].

3.1.1 Relation to $\mathcal{N} = 2^*$ A_1 system

Inspired by the appearance of a Jacobi polynomial said above, we strongly expect that its reduction to A_1 -Jack (or Gegenbauer) polynomials by further constraining three μ 's can be given a physical interpretation¹⁵. Namely, having in mind that an A_1 -Jack polynomial gets closely related to a specialized hypergeometric function

$${}_2F_1(-A, A + 2B, B + \frac{1}{2}; x) \tag{3.7}$$

and is the eigenstate of A_1 -type Calogero-Sutherland model, a limiting case of A_1 -type Calogero-Moser model as $p = \exp(2\pi i\tau)$ of Weierstrass's \wp -function goes to zero (or $\tau \rightarrow i\infty$), we cannot help suspecting that the constraint imposed on three μ 's leading to (3.7) should result from a one-punctured *pinched* torus. In other words, one may think of (3.7) as a two-point conformal block $\mathcal{B}[C_{1,2}]$ with one insertion being $\Phi_{1,2}$ defined on a pinched torus. Notice that redefining $x \sim \exp(i\lambda) \in \mathbb{C}^*$ makes the periodicity $\lambda \sim \lambda + 2\pi$ explicit.

To carry out the check, one also needs to know the degenerating process: $\mathcal{T}_{1,1} \rightarrow \mathcal{T}_{0,3}$. Given the fact that in the physical $\mathcal{T}_{0,3}$ theory four free hypermultiplets have their masses $\mu_1 \pm \mu_2 \pm \mu_4$ yielded from assigned momenta of three inserted Liouville primary fields [5], plausibly μ 's will now not be independent because in the former there are only two independent variables (\mathbf{a}, \mathbf{m}) , i.e. $\mathcal{N} = 2^*$ $SU(2)$ Coulomb branch parameter and adjoint hypermultiplet mass. Further, from a toric diagram associated with a Calabi-Yau three-fold engineering $\mathcal{N} = 2^*$ $SU(2)$ theory, one is able to read off masses of four free hypermultiplets in terms of (\mathbf{a}, \mathbf{m}) . Then, the constraint for μ 's gained from comparing (3.5) with (3.7) can be directly contrasted with what is derived above via an $\mathcal{N} = 2^*$ toric diagram.

An even interesting direction is to consider connections between various orthogonal polynomials by means of AGT picture. Serving as eigenstates of distinct Schrödinger equations (or two-body

¹⁴ $\epsilon_1 = -\epsilon_2$ serves as the *genus*-expansion parameter inside $Z_{\text{inst}} = \exp(\mathcal{F})$ since $\mathcal{F} = -\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_0 + \dots$ is referred to as the A-model topological string free energy w.r.t. a responsible Calabi-Yau three-fold in Type IIA theory.

¹⁵I thank Hirotaka Irie, Yutaka Matsuo and Akitsugu Miwa with whom I have communicated about these stuffs.

integrable systems), they are nonetheless transformed to one another by performing some limit which may acquire suitable geometric meaning in terms of Riemann surfaces. We wish to report these topics in an upcoming paper.

3.2 Triality and Kummer's 24 solutions

Mathematically speaking, multiplying prefactors like $\lambda^A(1-\lambda)^B$ just brings ${}_2F_1(a, b, c; \lambda)$ to another Riemann scheme containing solutions like ${}_2F_1(a', b', c'; \lambda)$ and so on. To say which scheme is more preferable seems not so essential. We decide to exclude these prefactors below also because in [23] this choice of \mathcal{B} did reproduce the gravitationally-corrected asymptotically-free SW prepotential \mathcal{F}_0 .

By applying AGT dictionary (1.3) together with (1.1), (1.2) and (3.6) specialized at $\beta = 1$, it is seen that other five of Kummer's twenty-four solutions can be generated from ${}_2F_1(a, b, c; \lambda)$ by elements $\{S, T, TS, STS, ST\}$:

(1) S : $\mu_1 \leftrightarrow \mu_2$

$$\begin{cases} a \rightarrow a \\ b \rightarrow b \\ c \rightarrow a + b - c + 1 \\ \lambda \rightarrow 1 - \lambda \end{cases}$$

(2) T : $\mu_4 \leftrightarrow \mu_2$

$$\begin{cases} a \rightarrow -a + c \\ b \rightarrow b \\ c \rightarrow c \\ \lambda \rightarrow \frac{\lambda}{\lambda - 1} \end{cases}$$

(3) TS : $(\mu_1, \mu_2, \mu_4) \rightarrow (\mu_2, \mu_4, \mu_1)$

$$\begin{cases} a \rightarrow b - c + 1 \\ b \rightarrow b \\ c \rightarrow a + b - c + 1 \\ \lambda \rightarrow \frac{\lambda - 1}{\lambda} \end{cases}$$

(4) STS : $\mu_4 \leftrightarrow \mu_1$

$$\begin{cases} a \rightarrow b + c - 1 \\ b \rightarrow b \\ c \rightarrow -a + b + 1 \\ \lambda \rightarrow \frac{1}{\lambda} \end{cases}$$

(5) ST : $(\mu_1, \mu_2, \mu_4) \rightarrow (\mu_4, \mu_1, \mu_2)$

$$\begin{cases} a \rightarrow -a + c \\ b \rightarrow b \\ c \rightarrow -a + b + 1 \\ \lambda \rightarrow \frac{1}{1-\lambda} \end{cases}$$

Finally, all of them are collected below:

$$\begin{cases} \text{(I)} \ {}_2F_1(a, b, c; \lambda) \\ \text{(II)} \ {}_2F_1(a, b, a + b - c + 1; 1 - \lambda) \\ \text{(III)} \ (1 - \lambda)^{-b} {}_2F_1(c - a, b, c; \frac{\lambda}{\lambda - 1}) \\ \text{(IV)} \ \lambda^{-b} {}_2F_1(b - c + 1, b, a + b - c + 1; \frac{\lambda - 1}{\lambda}) \\ \text{(V)} \ \lambda^{-b} {}_2F_1(b - c + 1, b, b - a + 1; \frac{1}{\lambda}) \\ \text{(VI)} \ (1 - \lambda)^{-b} {}_2F_1(-a + c, b, b - a + 1; \frac{1}{1 - \lambda}) \end{cases}$$

Although strictly speaking (III)–(V) have been dressed by either λ^b or $(1 - \lambda)^b$, it can merely be added by hand in order to preserve the given Riemann scheme. As a matter of fact, according to [64] we see that

$$\left\{ \begin{array}{ll} \text{(I)} \text{ (III)} & \lambda = 0 \text{ basis,} \\ \text{(II)} \text{ (IV)} & \lambda = 1 \text{ basis,} \\ \text{(V)} \text{ (VI)} & \lambda = \infty \text{ basis,} \end{array} \right.$$

where by “basis” we mean spanning a basis of solutions around there. Therefore, triality plays the role of interchanging solutions around three regular singularities $(0, 1, \infty)$. This manipulation thus manifests how triality is just understood in terms of another mathematical object—hypergeometric function.

4 Discussion

(I) Crossing symmetry and triality

Correlators Ω of some rational CFT including Liouville field theory (LFT) defined on a four-punctured Riemann sphere are endowed with the *crossing symmetry*. Usually, this fact is expressed as follows:

$$\Omega^{(4)} \left(\begin{array}{cc} \mu_2 & \mu_3 \\ \mu_4 & \mu_1 \end{array} \middle| \lambda \right) = \Omega^{(4)} \left(\begin{array}{cc} \mu_1 & \mu_3 \\ \mu_4 & \mu_2 \end{array} \middle| 1 - \lambda \right) \quad (4.1)$$

where $\Omega^{(4)}$ is analytic over $\mathbb{P}_*^1 : \mathbb{P}^1 \setminus \{0, 1, \infty\}$. When it comes to conformal blocks \mathcal{B} (main component constituting the above $\Omega^{(4)}$) though, the crossing symmetry is respected to a *covariant* extent:

$$\mathcal{B}_{\mu_{31}}^s \left(\begin{array}{cc} \mu_2 & \mu_3 \\ \mu_4 & \mu_1 \end{array} \middle| \lambda \right) = \int_{\frac{\mathbb{Q}}{2} + i\mathbb{R}^+} d\mu_{23} F_{\mu_{31}\mu_{23}}^L \left(\begin{array}{cc} \mu_2 & \mu_3 \\ \mu_4 & \mu_1 \end{array} \right) \mathcal{B}_{\mu_{23}}^t \left(\begin{array}{cc} \mu_1 & \mu_3 \\ \mu_4 & \mu_2 \end{array} \middle| 1 - \lambda \right). \quad (4.2)$$

Note that $F_{\mu_{31}\mu_{23}}^L$ specifies the invertible fusion matrix in between s - and t -channel LFT conformal blocks. When one of four primary fields gets degenerate then \mathcal{B} reduces to a hypergeometric function such that (4.2) reads

$$\begin{aligned} {}_2F_1(a, b, c; \lambda) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b-c+1; 1-\lambda) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b, c-a-b+1; 1-\lambda) \end{aligned}$$

as having been indicated by connection coefficients listed in (2.2).

Notice that (I)-(VI) collected in Sec. 3.2 is somewhat unexpected and independent of (2.2). One can just hardly regard (2.2) as any evidence showing triality (or S -duality) as AGT conjecture relates Liouville conformal blocks to Nekrasov's instanton partition functions. This is because the latter never splits into two compatible (and linearly independent) components. On the other hand, given certain Riemann scheme dictated by four flavor masses, the permutation between six Kummer's solutions upon imposing triality ultimately amounts to another geometric realization of \mathbf{S}_3 outer automorphism of \mathbb{P}_*^1 . This picture will never be obtained by merely applying the crossing symmetry though.

(II) Geometric realization of triality

In order to make clearer the above statement, let us first write down the following expansion:

$$\Psi(\lambda) = {}_2F_1\left(\frac{a}{\hbar}, \frac{b}{\hbar}, \frac{c}{\hbar}; \lambda\right) = \exp\left(-\left(\hbar^{-1}W_0 + W_1 + \hbar W_2 + \cdots\right)\right).$$

The zeroth-order term W_0 is obtained by means of the WKB approximation

$$\exp(-\hbar^{-1}W_0) = \exp\left(i\hbar^{-1} \int^\lambda dz \sqrt{T(z)}\right)$$

w.r.t. the Q -form $\left(\hbar^2 \partial_\lambda^2 + T(\lambda)\right)\Psi(\lambda) = 0$ of an usual hypergeometric equation¹⁶. It has been found in [10] that

$$\begin{aligned} W_0 &= \int^\lambda dz \sqrt{\frac{-a_1^2 z(1-z) + a_2^2(1-z) + a_3^2 z}{z^2(1-z)^2}}, \\ a &= -a_1 + a_2 - a_3, \quad b = a_1 + a_2 + a_3, \quad c = 2a_2. \end{aligned}$$

In addition, the curve

$$\mathcal{P} : y^2 = T(z) = \frac{-a_1^2 z(1-z) + a_2^2(1-z) + a_3^2 z}{z^2(1-z)^2}$$

turns out to be dual to \mathbb{P}_*^1 [10, 28] since it stands for the real two-dimensional portion of M5-branes occupying $\mathbb{R}^{1,3} \times \mathcal{P}$ which arise from the uplift of a type IIA brane configuration (hence $\epsilon_1 + \epsilon_2 = 0$) engineering $\mathcal{T}_{0,3}(A_1)$ theory of four free hypermultiplets. Therefore, choosing one Riemann scheme means fixing all parameters of \mathcal{P} (pants diagram), and interchanging solutions to its corresponding Q -form is equivalent to manifesting the outer automorphism \mathbf{S}_3 of \mathcal{P} .

In summary, \mathbf{S}_3 originally stemming from the outer automorphism of $spin(8)$ (as seen from

¹⁶ \hbar is restored momentarily for computational convenience during applying the WKB method.

its Dynkin diagram) at the low-energy regime has been once interpreted as $\mathbf{S}_4/\mathbb{Z}_2 \times \mathbb{Z}_2$ built on a four-punctured sphere via AGT dictionary (1.3). Through studying (1.1) and (1.2) under the hypergeometric nature of this sort of $SU(2)$ instanton partition functions, we see the revival of \mathbf{S}_3 in a fashion naturally encoded in \mathcal{P} dual to \mathbb{P}_*^1 . This presentation dose provide a new geometric though simple viewpoint of (1.1) and (1.2) spelt out by Seiberg and Witten.

5 Summary

Let us briefly summarize our main results. The geometric origin of triality stemming from the outer automorphism group \mathbf{S}_3 of the quantum flavor symmetry $Spin(8)$ has long been pursuit. In Vafa’s F-theory setup, a D_4 -type singularity on an elliptically fibered K3 can be used to engineer an $\mathcal{N} = 2$ A_1 -type $N_f = 4$ SCFT due to an arbitrary string coupling. While Vafa’s picture compactified down to IIB theory stresses a geometric realization of u -plane parameterizing Coulomb branch, triality, namely (1.1) and (1.2), connecting physically equivalent theories seems not immediately visible. This is because now one is confined nearby a slightly deformed D_4 -type singularity whereas in addition to bare mass parameters (positions of D7-branes located on u -plane) triality involves further an asymptotic piece of information, say, τ_0 at $u \rightarrow \infty$. This problem of τ_0 can be once remedied if one notices a bijection between the “fundamental” domain $\mathbb{H}/\Gamma(2)$ of τ_0 and moduli space of four marked points on a Riemann sphere by means of the celebrated λ -function (2.10). The latter object denoted as $C_{0,4}$ emerges in Gaiotto’s revolutionary description of an $\mathcal{N} = 2$ $SU(2)$ $N_f = 4$ SCFT. Instead, how to encode mass transformation rules into $C_{0,4}$ now turns out to be invisible.

What comes to one’s rescue is AGT conjecture which states precisely (3.3). Equipped with it, (1.1) and (1.2) performed onto bare masses contained in Z_{inst} as well as τ_0 are then translated into interchanging six hypergeometric functions belonging to three regular singularities under certain Riemann scheme, provided one primary insertion of the four-point spherical conformal block gets degenerate. These arguments do provide another insight into capturing triality geometrically, e.g. permutation around vertices of a Schwarz triangle. Note that solutions in (3.8) are not equal to one another echoes the fact that $\mathcal{B}[C_{0,4}]$ along is basically not S -duality invariant or Nekrasov’s partition function on \mathbb{R}^4 transforms nontrivially under S -duality as stressed in [46].

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